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On the Universal Korovkin Closure of Subsets in Vector Lattices

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INTRODUCTION

In [13] we introduced the notion of the universal Korovkin closure $\mathfrak{K}_u(M)$ of a subset M in an arbitrary locally convex vector lattice E (see 1.2 of the present paper for definition). M is called a universal Korovkin system if $\mathfrak{K}_u(M)$ equals E.

In our opinion there are several interesting aspects of this concept. First it is implicitly known, that an I-Korovkin system in the space $C(X)$ of all real-valued continuous functions on a compact T_2-space X is always universal (cf. 1.6). (A subset M of E is called an I-Korovkin system if it satisfies the following condition: an equicontinuous net of positive operators converges to the identity, pointwise on E, whenever it converges to it pointwise on M).

Another reason for considering universal Korovkin systems is the fact that the image of such a system under a continuous lattice homomorphism with a dense range is again a universal Korovkin system (see 1.7). This yields an easy method for constructing these systems. In fact most of the I-Korovkin systems already known in special function lattices turn out to be universal (see sect. 1.8 for details and examples).

Finally we shall show in a forthcoming paper [16] that in many function lattices the universal Korovkin closure and the I-Korovkin closure of a large class of subsets coincide. A fortiori the same holds for the corresponding Korovkin systems (as is the case in $C(X)$).

It is the main purpose of the present paper to characterize the universal Korovkin closure for a wide class of subsets in locally convex vector lattices thus generalizing the results of [13]. This will be done in §1. The second paragraph is devoted to the proofs of the main results of §1. They are based on an abstract disintegration theorem which may be of interest for its own sake. As suggested by the work of K. Donner ([3, 4]) we discuss in §3 some of the consequences which occur if one eliminates the condition of equicontinuity of the nets in "Korovkin's theory". In fact we show that under

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special but nevertheless reasonable circumstances this can be done iff the underlying space is of type $C(X)$ ($X \subset \mathbf{R}^n$ compact, n suitable). This is a generalization of [14].

1. The Universal Korovkin Closure

Though many applications are concerned with very concrete function spaces, we get the most transparent and striking results, if we use the set-up of the theory of locally convex vector lattices (see [8, 10, 17] for details).

Let E be a fixed locally convex vector lattice, let M be an arbitrary subset of E, and L_M *the closed linear hull of* M. Let $M^\wedge$ be the set of all infima of finite, nonempty subsets of L_M, i.e. $M^\wedge = \{\inf(A): \varnothing \neq A \subset L_M, A \text{ finite}\}$, and let C_M denote the closure of $M^\wedge$.

Definition 1.1 ([13, 1.1]). The closed linear subspace $\mathfrak{H}_M := C_M \cap (-C_M)$ is called the subspace of *M-harmonic elements*. In other words, an element x is M-harmonic if to each neighborhood U of O there exist elements $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ in L_M satisfying $x - \inf(u_1, \ldots, u_m) \in U$ and $x - \sup(v_1, \ldots, v_n) \in U$.

The *universal Korovkin closure* can most easily be defined in the following manner:

We consider triples $\mathfrak{T} = (F, S, (T_\alpha))$, where F denotes another locally convex lattice, S a continuous linear lattice homomorphism and (T_α) an equicontinuous net of positive linear mappings from E to F. The set $\mathfrak{S}(\mathfrak{T}) = \{y \in E: \lim T_\alpha y = Sy\}$ is called *quasi-shadow of* $\mathfrak{T}$. The quasi-shadows of all triples build up a subclass of the power set of E, therefore there exists their intersection.

Definition 1.2 ([13, 2.2]). Let M be a subset of E. The intersection of all quasi-shadows containing M is called *universal Korovkin closure* (sometime *total shadow*) $\mathfrak{K}_u(M)$ of M, in symbols: $\mathfrak{K}_u(M) = \bigcap_{M \subset \mathfrak{S}(\mathfrak{T})} \mathfrak{S}(\mathfrak{T})$. M is called a *universal Korovkin system*, if $\mathfrak{K}_u(M)$ equals E.

Trivially any universal Korovkin system is in particular an I-Korovkin system. In [13, Thm. 1.2] we showed (under more general assumptions on E) that $\mathfrak{H}_M$ *is always contained in* $\mathfrak{K}_u(M)$; special cases of this result were published in the meantime independently by Fakhouri [5], and Kitto and Wulbert [7].

We call a linear subspace L of E to be *positively generated*, if every $z \in L$ is the difference $z = x - y$ of two nonnegative elements x and y of L. A closed linear subspace L is called *weakly positive* if it contains a positively generated, dense, linear subspace.

Now we are able to formulate our main results (Theorem 1.3 and its Corollary).

THEOREM 1.3. *Let E be a locally convex vector lattice and M a subset of E, the closed linear hull of which is weakly positive. For an element x_0 of E the following statements are equivalent*:

(a) $x_0 \notin \mathfrak{K}_u(M)$

(b) $x_0 \notin \mathfrak{H}_M$

(c) *There exists a measure space $(\Omega, \Sigma, \lambda)$, and a continuous linear lattice homomorphism R from E into $F = L^1(\Omega, \Sigma, \lambda)$, furthermore a continuous positive linear mapping T from E into F, such that Tz equals Rz for all $z \in M$ (a fortiori for all $z \in \mathfrak{H}_M$), but Tx_0 is different from Rx_0.*

COROLLARY 1.4. *Let E and M be as above. Then the space $\mathfrak{H}_M$ of all M-harmonic elements is equal to the universal Korovkin closure $\mathfrak{K}_u(M)$. In particular M is a universal Korovkin system iff $\mathfrak{H}_M$ equals E.*

REMARKS AND EXAMPLES 1.5. (1) The results above were proved in [13] under stronger and more complicated conditions for M.

(2) If E is separable, we can choose in (c) of 1.3 the standard Lebesgue measure space $([0, 1], \lambda)$.

(3) As we shall show in a forthcoming paper [16] in many cases one can use $F = \mathbf{R}$ instead of a general L^1-space (as is the case for $E = C(X)$).

(4) As pointed out above (see the section after 1.2), the relation $\mathfrak{H}_M \subset \mathfrak{K}_u(M)$ holds for arbitrary subsets M. Thus M is a universal Korovkin system if $\mathfrak{H}_M$ equals E. The difficulty is to show the reverse implication. However in practice it is often quite easy to prove $\mathfrak{H}_M$ to be equal to E. This holds obviously if any element y of a dense linear subspace G of E is the weak limit of the net $\{x \in M^\wedge : y \leqslant x\}$ (use [10, V.4.3]; for a detailed discussion see [13, §3]). As an application we give the following example: let E be the space of all continuous real-valued functions on $\mathbf{R}$, vanishing at $\pm\infty$. Equipped with the usual structure and the sup-norm E is a Banach lattice. E. Scheffold [12] proved by other methods that the set $M = \{\exp(-x^2), x\exp(-x^2), x^2\exp(-x^2)\}$ is an I-Korovkin system in E. Now we show without using this result that M is in fact a universal Korovkin system. Let A denote the space of all polynomials of degree at most two. If g is a twice continuously differentiable function with a compact support an elementary calculation shows that the equation $g(x) = \inf\{h(x) : h \in A, g \leqslant h \text{ on } \mathbf{R}\}$ holds pointwise on $\mathbf{R}$. Multiplying both sides by $\exp(-x^2)$ and applying Dini's theorem (to the two-point compactification of $\mathbf{R}$) we get $\mathfrak{H}_M = E$.

The rest of this paragraph is devoted to the ideas of universal Korovkin systems mentioned in the introduction. The first lemma is a reformulation of known facts.

LEMMA 1.6. *Let E be the space $C(X)$ of all real-valued continuous functions on a compact T_2-space X. Then any I-Korovkin system of E is universal.*

Proof. If the polar L^0 of $L := L_M$ contains a $\mu > 0$, then the operator $T = I + 1_X \otimes \mu$ shows M not to be an I-Korovkin system. Thus we have $L^0 \cap E_+' = \{0\}$, and therefore $E = \overline{L - E_+}$ by the bipolar theorem. Since especially 1_X can be approximated uniformly on X by elements of $L - E_+$, L must contain a strict positive function f_0 (i.e. $f_0(x) > 0$ for all x in X). Since M is an I-Korovkin system, we now conclude (similarly to [1]) that every g in E is equal to $\hat{g}: x \to \inf\{h(x): g \leqslant h \in L\}$ which implies $\mathfrak{H}_M$ to be equal to E (cf. [13, 3.6]).

The usefulness of the notion of a universal Korovkin system is now illustrated by the following proposition and the applications below.

PROPOSITION 1.7. *Let E and F be locally convex vector lattices, and let T be a continuous lattice homomorphism from E to F with a dense range. If M is a universal Korovkin system in E, then $T(M)$ is a universal Korovkin system in F.*

The proof follows quite easily from the definition.

APPLICATIONS 1.8. (1) Let $E = C(X)$ be as in 1.6, and let M be a universal Korovkin system in E. If f is a strictly positive function then $f \cdot M = \{f \cdot g: g \in M\}$ again is a universal Korovkin system (consider the lattice isomorphism T defined by $Tg = f \cdot g$).

(2) Let F be a locally convex vector lattice containing $E = C(X)$ (X compact) as a dense sublattice. Then the canonical embedding of E into F is continuous. Hence every I-Korovkin system M in $C(X)$ is a universal Korovkin system in F (use 1.6 and 1.7). For example, if X is in $\mathbf{R}^n$, then $\{1, x_1, ..., x_n, x_1^2, ..., x_n^2\}$ is a universal Korovkin system in all Banach function spaces $L^\rho(X, \Sigma, \mu)$ for which ρ is an order continuous function norm, in particular in every $L^p(X, \Sigma, \mu)$ $(1 \leqslant p < \infty)$.

(3) Let F be the space of all continuous real-valued functions on $[0, \infty)$, vanishing at infinity, and let F be equipped with the sup-norm. Then $M = \{\exp(-x),\ x\exp(-x),\ x^2\exp(-x)\}$ is a universal Korovkin system in F. To see this consider $E = C([0, 1])$ and the operator T defined by

$$(Th)(x) = e^{-x}(1 + x)^2\, h((1 + x)^{-1})(0 \leqslant x < \infty).$$

Then L_M is the image of the linear hull of the classical Kovovkin system $\{1, x, x^2\}$ (E. Scheffold [12] proved M to be an I-Kovovkin system by other methods.).

(4) Let Y be a locally compact space, let G be a sup-norm dense lattice ideal of $C_0(Y)$ (the space of all real-valued continuous functions vanishing at infinity), and let F be an arbitrary locally convex vector lattice, containing G as a dense sublattice.

Suppose there exist a continuous embedding Ψ of Y into a compact space X, and a strict positive function f_0 in G. If for $h \in C(X)$, Th is defined by $(Th)(y) = f_0(y)\, h(\Psi(y))$, and if M is an I-Korovkin system in $C(X)$, then $T(M) = \{f_0 \cdot g \circ \Psi : g \in M\}$ is a universal Korovkin system in F.

EXAMPLE. Let $F = l^p(\mathbf{N} \times \mathbf{N})$ ($1 < p < \infty$, $\mathbf{N} \times \mathbf{N}$ equipped with the discrete topology). Set $f_0(k, n) = (kn)^{-1}$ and $\Psi(k, n) = (1/k, 1/n) \in [0, 1]^2$. Then if $f_1(k, n) = (k^2n)^{-1}$, $f_2(k, n) = (k^3n)^{-1}$, $f_3(k, n) = (kn^2)^{-1}$, $f_4(k, n) = (kn^3)^{-1}$, $M = \{f_0, ..., f_4\}$ is a universal Korovkin system in F. Let however Ψ be a bijection of $\mathbf{N} \times \mathbf{N}$ onto the rationals in $[0, 1]$. Then if $g_1(k, n) = (kn)^{-1} \cdot \Psi(k, n)$. $g_2(k, n) = (kn)^{-1} \cdot (\Psi(k, n))^2$, $M = \{f_0, g_1, g_2\}$ is a smaller universal Korovkin system in F (cf. [13, Theorem 3.3]).

2. An Abstract Disintegration Theorem and the Proofs of the Results of 1

To prove the disintegration theorem (2.4 below) we first recapitulate some ideas of [15] in order to make the present paper self-contained.

Let G and H be real vector spaces. A real-valued function on $G \times H$ is called a *bisublinear form* if it is separately sublinear in both variables. Such a bisublinear form q is called *projective* if $0 \leqslant \Sigma q(x_i, g_i)$ holds whenever $\Sigma x_i \otimes g_i$ ($\in G \otimes H$) is equal to zero. An easy calculation proves

LEMMA 2.1 ([15, Lemma 1]). *For a sublinear form q the following two statements are equivalent*:

(a) *q is projective*

(b) *$\bar{q}$, defined on $G \otimes H$ by $\bar{q}(w) = \inf\{\Sigma q(x_i, g_i): w = \Sigma x_i \otimes g_i\}$, is finitely-valued and sublinear.*

This lemma yields

PROPOSITION 2.2 ([15, Satz 1]). *Let x' be a linear form on G, let u be a fixed element of H, and let q be a projective bisublinear form on $G \times H$*

satisfying $\langle x, x' \rangle \leq \bar{q}(x \otimes u)$ *for all* x *in* G. *Then there exists a linear mapping* T *from* H *into the algebraic dual* G^* *of* G *satisfying*

(i) $\langle x, Tg \rangle \leq q(x, g)$ *for all* $(x, g) \in G \times H$, *and*

(ii) $Tu = x'$.

Proof. We define a linear form ρ_0 on the subspace $G \otimes u$ of $G \otimes H$ by setting $\rho_0(x \otimes u) = \langle x, x' \rangle$. ρ_0 is easily seen to be dominated by $\bar{q}$, constructed from q as in 2.1. Now extend ρ_0 to a linear form ρ on $G \otimes H$, again dominated by $\bar{q}$. The desired mapping T then is given by the relation $\langle x, Tg \rangle = \rho(x \otimes g)$.

We now have to recall some basic facts about AL-spaces.

2.3. A Banach lattice F is called *AL-space* if its norm is additive on the positive cone $F_+ = \{y \in F: y \geq 0\}$.

(a) Let F be an AL-space. Then by Kakutani's representation theorem ([11, proof of II. 8.5, p. 115]) there exists a topological direct sum X of compact spaces and a strict positive Radon-measure λ on X, such that F is isometrically and norm isomorphic to the space $L^1(X, \lambda)$.

(b) (s. [11, III. 5.3, p. 171]) We can choose X in such a way, that in every equivalence class $\hat{f} \in L^1(X, \lambda)$ there exists exactly one continuous function f from X into $\bar{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$, attaining the values $\pm\infty$ only on closed, disjoint λ-null sets. The subset $\hat{F}$ of $C(X, \bar{\mathbf{R}})$ $(=\{g \in \bar{\mathbf{R}}^X: g \text{ is continuous}\})$, consisting of those elements described above, builds up a vector lattice (with respect to the usually defined operations a.e.); moreover, if f is in $\hat{F}$, and $g \in C(X, \bar{\mathbf{R}})$ such that $0 \leq |g| \leq |f|$, then g is in $\hat{F}$; in particular the space $C_{00}(X)$ of all real-valued continuous functions with compact support is a lattice ideal in F, which is dense with respect to the norm $f \to \int |f| \, d\lambda =: \|f\|_1$. From now on we identify $L^1(X, \lambda)$ with $\hat{F}$.

(c) If we choose X as in Sect. (b) the dual space F' can be identified with the space $C_b(X)$ of all real-valued, continuous, bounded functions on X; this can and will be done in such a way that the canonical bilinear form on $F \times F'$ is transformed into $\langle f, g \rangle = \int fg \, d\lambda$ (for all $f \in \hat{F}$, $g \in C_b(X)$).

(d) Let ρ be a strictly positive linear form on $\hat{F}$ (i.e. $|f| \neq 0$ always implies $\langle |f|, \rho \rangle \neq 0$); its restriction (also denoted by ρ) to $C_{00}(X)$ is a Radon-measure on X, absolutely continuous with respect to λ; moreover, the mapping $V_\rho : f \to f \cdot \rho(\langle g, f \cdot \rho \rangle = \int fg \, d\rho)$ sends $\hat{F}$ onto a lattice ideal of the norm dual $C_b(X)'$ of $C_b(X)$.

We now are able to present the main result of this section.

THEOREM 2.4. *Let* G *be an arbitrary vector space over* $\mathbf{R}$, *and* F *be an* AL*-space. Let* ρ *be a positive linear form on* F, x' *a linear form on* G *and* Q

a sublinear mapping from G into F, satisfying $\langle x, x' \rangle \leqslant \langle Q(x), \rho \rangle$ for all $x \in G$. Then there exists a linear mapping S from G into F, dominated by Q (i.e. satisfying $Sx \leqslant Q(x)$ for all $x \in G$) such that $\langle x, x' \rangle = \langle Sx, \rho \rangle$ for all $x \in G$.

REMARKS. (1) The proof of this theorem can be easily extended to the case where F is an arbitrary Banach lattice with an order continuous norm. There is, however, no need for such a generalization within the present context.

(2) The theorem may be used to generalize the concrete disintegration theorem of Strassen ([6, p. 100]) in a way similar to one we used already in a special case ([15, Theorem 3]).

Proof of 2.4. 1*st step*: We reduce 2.4 to the case where ρ is strictly positive. Suppose for the moment the theorem has been proved in this special case. Consider the band $J = \{x \in F: \langle |x|, \rho \rangle = 0\}$. Let P be the canonical projection from F onto J, and $I - P = R$ be the projection onto $J^\perp = \{y \in F: \inf(|x|, |y|) = 0 \text{ for all } x \text{ in } J\}$ (cf. [11, p. 113]). PQ and RQ are sublinear mappings from G into the AL-spaces J, and $J^\perp$, respectively. By the generalized theorem of Hahn-Banach ([11, p. 109]) there exists a linear mapping $S_1 \leqslant PQ$ from G into J. ρ is strictly positive on $J^\perp$, and we have $R'\rho = \rho$, $P'\rho = 0$, hence $\langle x, x' \rangle \leqslant \langle RQ(x), \rho \rangle$. Also if S_2 is a mapping from G into $J^\perp$, satisfying $S_2 \leqslant RQ$ and $\langle x, x' \rangle = \langle S_2 x, \rho \rangle$, then $S = S_1 + S_2$ is the desired linear mapping, because we have $S_1 \leqslant PQ$ and therefore $\langle \pm S_1 x, \rho \rangle \leqslant \langle PQ(x) + PQ(-x), \rho \rangle = 0$, hence $\langle S_1 x, \rho \rangle = 0$ for all x in G.

2*nd step*: Now let ρ be strictly positive, and identify F with $\hat{F}$, F' with $C_b(X) =: H$, and ρ with the corresponding Radon measure on X, as in 2.3.

We shall construct a projective bisublinearform q on $G \times H$ by setting $q(x, g) = \int Q(g(z)\, x)(z)\, d\rho(z)$. Because of $Q(g(z)\, x) = g^+(z)\, Q(x) + g^-(z)\, Q(-x)$ the right-hand side of the equality exists ($g^+ = \sup(g, 0)$, $g^- = -\inf(g, 0)$).

If g is kept fixed, $q(\cdot, g)$ is sublinear because ρ is isotone and Q is sublinear. For fixed x we get $Q((g_1(z) + g_2(z))\, x) = Q(g_1(z)\, x + g_2(z)\, x) \leqslant Q(g_1(z)\, x) + Q(g_2(z)\, x)$ for every pair $g_1, g_2 \in H$ and every $z \in X$. This implies $q(x, \cdot)$ to be sublinear on H; so q has been shown to be bisublinear. Now let $\sum_{i=1}^n x_i \otimes g_i$ ($\in G \otimes H$) be equal to zero. For fixed $z \in X$ we get $0 = \Sigma g_i(z)\, x_i$ hence $0 \leqslant \Sigma Q(g_i(z)\, x_i)$, hence $0 \leqslant \Sigma q(x_i, g_i)$ which proves q to be projective; similarly, one gets $q(x, 1_x) = \bar{q}(x \otimes 1_x)$.

3*rd step*: We now construct S with the aid of q from step 2. Setting $u = 1_X$ (i.e. the constant function one) we get by 2.2 a linear mapping T from H into the algebraic dual G^* of G satisfying

(i) $\langle x, Tg\rangle \leqslant q(x, g)$ and

(ii) $T1_X = x'$. The restriction $\bar{S}$ onto G of the adjoint mapping T^* (from G^{**} into H^*) satisfies $\langle g, \bar{S}x\rangle \leqslant q(x, g) = \int Q(g(z)\,x)(z)\,d\rho(z) \leqslant \int |g(z)|\,(|Q(x)|(z) + |Q(-x)|\,(z))\,d\rho(z) =: A(x, g)$ for all $g \in H$, $x \in G$. The same argument yields $-\langle g, Sx\rangle = \langle g, S(-x)\rangle \leqslant A(x, g)$. This implies that with respect to the order on $C_b(X^*)$ induced by the positive cone of the norm dual $C_b(X)'$ the absolute value of $\bar{S}x$ exists and is dominated by $(|Qx| + |Q(-x)|)$. ρ, i.e. $\bar{S}x$ is an element of $V_\rho(\hat{F})$. Since V_ρ is a lattice isomorphism (see Sect. (d) of 2.3), $S = V_\rho^{-1} \circ \bar{S}$ is easily seen to be a mapping with the required properties. This completes the proof.

Proof of 1.3 (cf. [13, p. 104, proof of 2.5]). It is enough to prove (b) $\Rightarrow$ (c) because of the remark after 1.2.

1st step: Let $x_0 \notin \mathfrak{H}_M$ be fixed, and w.l.o.g. assume x_0 not to be in C_M. Then there exists a continuous linear form μ in the polar $C_M{}^0$ of C_M satisfying $\langle x_0, \mu\rangle > 0$. Let ν be equal to $|\mu|$ and let $p(z) = \langle |z|, \nu\rangle$ be the seminorm, determined uniquely by ν. The completion F of the quotient space $E/p^{-1}(0)$ with respect to the induced norm $\|\cdot\|_1$ is an AL-space containing $E/p^{-1}(0)$ as a dense sublattice. The quotient mapping R is a continuous linear lattice homomorphism, and $\overline{R(L_M)} = L_{R(M)}$ is weakly positive (as L_M is by assumption). Furthermore, we have $\overline{R(C_M)} = C_{R(M)}$ (built up in F), and the (uniquely determined) continuous extension of μ (also denoted by μ) is contained in the polar $C^0_{R(M)}$ of $C_{R(M)}$ whence Rx_0 is not in $C_{R(M)}$. Since $L_{R(M)}$ is weakly positive, it contains a dense linear subspace L which is positively generated. Thus, $G = \{y \in F\colon \exists u, v \in L \text{ with } u \leqslant y \leqslant v\}$ is a lattice ideal as well as its closure $\bar{G}$, which contains $L_{R(M)}$ and obviously contains $C_{R(M)} = C_L$. We set $\rho = \mu^-$ ($= \sup(-\mu, 0)$) and for $y \in G$ we define $Q(y)$ to be equal to $\inf\{z \in C_{R(M)} : y \leqslant z\}$.

The subset, of which the infimum is taken, is not empty, directed downwards, and bounded from below. Because of the closedness of $C_{R(M)}$ and the order continuity of the norm, $Q(y)$ has to be in $C_{R(M)}$. Let $x' \in G^*$ be the restriction of $\mu^+ = \sup(\mu, 0)$ to G. From $\langle z, \mu^+\rangle \leqslant \langle z, \mu^-\rangle$ for all z in $C_{R(M)}$ we have $\langle x, x'\rangle = \langle x, \mu^+\rangle \leqslant \langle Q(x), \mu^+\rangle \leqslant \langle Q(x), \mu^-\rangle$. By 2.4 we get a linear mapping S from G into F satisfying $Sx \leqslant Qx$ for all $x \in G$ as well as $\langle x, \mu^+\rangle = \langle Sx, \mu^-\rangle$. Because of $Q(x) \leqslant 0$ for all $x \leqslant 0$, S has to be positive (with respect to the order induced on G by F).

2nd step: Set $J = \{y \in F\colon \langle |y|, \mu^+\rangle = 0\}$, let P_1 be the band projection onto J, $P_2 = I - P_1$ the band projection onto $J^\perp$ (cf. the 1st step in the proof of 2.4), and let T_0 be the well-defined positive linear operator $T_0 = P_1S + P_2$ from G into F. The relations $P_1'\mu^+ = P_2'\mu^- = 0$ and $P_1'\mu^- = \mu^-$, $P_2'\mu^+ = \mu^+$ yield at once $\langle T_0x, \mu^+\rangle = \langle T_0x, \mu^-\rangle = \langle x, \mu^+\rangle$, hence $\|T_0x\|_1 = \langle |T_0x|, \mu^+ + \mu^-\rangle \leqslant \langle T_0|x|, \mu^+ + \mu^-\rangle \leqslant 2\|x\|_1$, since $|x|$ is in G whenever x is.

T_0 is therefore continuous with respect to the norm on F and its restriction to G. Hence T_0 is uniquely extendable to a positive linear operator T_1 from $\bar{G}$ into F, satisfying $T_1'\mu^+ = T_1'\mu^- = \mu^+$ (restricted to $\bar{G}$).

3rd step: The closed ideal $\bar{G}$ is a band; therefore (s. [11, II. 8.3]) there exists a positive projection P from F onto $\bar{G}$. Let U be equal to T_1P and y an arbitrary element in L. Then $\pm y$ is in $G \cap C_{R(M)}$, hence $Q(\pm y) = \pm y$, and therefore $\pm U(y) = U(\pm y) = T_0(\pm y) = P_1S(\pm y) + P_2(\pm y) \leqslant P_1Q(\pm y) + P_2(\pm y) = \pm y$, i.e. $Uy = y$.

Furthermore we have $\langle URx_0, \mu^-\rangle = \langle URx_0, \mu^+\rangle$, hence $\langle URx_0, \mu\rangle = 0 \neq \langle Rx_0, \mu\rangle$. If now we identify F with its isomorphic representation as in Sect. (a) of 2.3 the mappings R and $T = UR$ are the desired ones. This completes the proof of 1.3.

Proof of 1.4. This follows at once from 1.3 and the relation $\mathfrak{H}_M \subset \mathfrak{K}_u(M)$ already mentioned.

Proof of Remark 2. Clear (cf. the corresponding proof in [13, p. 105, Sect. VI]).

3. A Korovkin-type Theorem for Nonequicontinuous Nets

So far we developped our theory only for equicontinuous nets of positive operators.

If we take, however, nonequicontinuous nets into our consideration (as was done by Šaškin [9], Bauer [1, 2], Donner [3, 4], and others) we are lead to the following definition.

Definition 3.1. *Let E be a locally convex vector lattice and M an arbitrary subset of E. The Korovkin closure $\mathfrak{K}_0(M)$ consists of all elements x having the following property: if (T_α) is an arbitrary net of continuous positive linear mappings from E into itself then* $\lim T_\alpha y = y$ *for all $y \in M$ always implies* $\lim T_\alpha x = x$. *M is called optimal Korovkin system, if $\mathfrak{K}_0(M)$ equals E.*

It is well-known ([13, 3.1]) that a *complete locally convex vector lattice possesses a finite universal Kovovkin system iff it is finitely generated.* This holds for example for all common function spaces. Unfortunately the same statement is no longer true, if we replace "universal" by "optimal", as Proposition 3.2, a generalization of the main result in [14], shows. Let us recall the following notion ([11, II. 1.8]): a vector lattice E is called relatively uniformly complete (r u c for short), if every principal ideal $E_y = \bigcup_{n\in\mathbf{N}}\{x \in E: |x| \leqslant ny\}$ $(y > 0)$ is complete with respect to the gauge p_y of the order intervall $[-y, y]$. For example, *every sequentially complete locally convex vector lattice is r u c.*

PROPOSITION 3.2. *Let E be a r u c, locally convex, barreled vector lattice. Then the following two statements are equivalent:*

(a) *E possesses a finite optimal Korovkin system.*

(b) *There exist a positive integer n and a compact subset X of* $\mathbf{R}^n$ *such that E is topologically and lattice isomorphic to the space C(X) of all real-valued continuous functions on X (equipped with the usual structures).*

Proof. (b) ⇒ (a): This is well-known ([9]).

(a) ⇒ (b): Let M be a finite optimal Korovkin system, and $u = \Sigma_{x\in M} | x |$. $M_1 = M \cup \{u\}$ again is such a system. By [4] $\mathfrak{K}_0(M_1)$ is contained in $L^b_{M_1} = \bigcup_{x,y\in L_{M_1}} [x, y]$, which obviously equals E_u (see above). By our assumption we get $E = \mathfrak{K}_0(M) \subset \mathfrak{K}_0(M_1) \subset E_u$, i.e. the closed order intervall $[-u, u]$ is absorbing, hence a barrel, and therefore a neighborhood of 0. Since on the other hand each neighborhood of 0 absorbs every order intervall, the given topology agrees with that one induced by the gauge p_u of $[-u, u]$. With respect to the latter E is complete by assumption, hence an AM-space with unit ([11, II. 7.2]) which in turn yields (E, p_u) to be fully isomorphic to $C(X)$ for a suitable compact space X [[11, II. 7.4]). Combining all the details proved so far and using Šaškin's result [9] we immediately get (b).

REMARKS AND EXAMPLES 3.3. (1) Example 3.2 in [13] tells us, that we cannot drop our assumption on E to be r u c, even if we replace $C(X)$ by a dense sublattice of such a space.

(2) As was observed by many other authors the set $M = \{x, x^2, x^3\}$ is not an optimal Korovkin system in any of the spaces $L^p([0, 1], \lambda)$ $(1 \leqslant p < \infty)$ or in $G = \{f \in C([0, 1]) : f(0) = 0\}$ (equipped with the sup-norm). This follows at once from 3.2. However, M is a universal Korovkin system in each of these spaces, as is easily shown by means of 1.7 (consider the lattice isomorphism T from $C([0, 1])$ into such a space defined by $(Th)(x) = xh(x)$).

(3) As a supplement to the proposition above we now give an example of a normed and r u c vectorlattice E which is not barreled, possesses a finite optimal Korovkin system, but which is not even lattice isomorphic (much less topological and lattice isomorphic) to a space $C(X)$ for any compact X in a finite-dimensional vector space.

Let E denote the subspace of $L^\infty([0, 1], \lambda) =: G$ consisting of all (equivalence classes of) Riemannian integrable functions equipped with the L^1-norm $\| \cdot \|_1$. By a well-known theorem of Carathéodory for each $g \in E$ we have $g = \inf\{f \in C([0, 1]) : g \leqslant f\} = \sup\{h \in C([0, 1]) : h \leqslant g\}$ (the inf and sup are taken in G). This easily yields $M = \{1, x, x^2\}$ to be an optimal Korovkin system in E (use [13, 1.5] or [4]). Since $(E, \| \cdot \|_\infty)$ is complete and contains a closed sublattice isomorphic to $l^\infty(\mathbf{N})$ and containing the constants, there is no $n \in \mathbf{N}$ and no compact subset of $\mathbf{R}^n$ such that E would be lattice isomorphic to $C(X)$ (else the Stone-Čech-compactification $\beta(\mathbf{N})$ of $\mathbf{N}$ would be a quotient space of X).

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